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ONE STEP HYBRID BLOCK METHODS FOR GENERAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS USING LAGUERRE POLYNOMIAL

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Abstract: In this paper, self-starting hybrid block method is proposed for the solution of general second order initial value problem of the form $Y' = f(x,y,y')$ directly without reducing it to first order of ODEs. The method was developed using Laguerre polynomial as basis function and the method was augmented by the introduction of off-step points in order to bring zero stability and upgrade the order of consistency of the method. An advantage of the derived continuous scheme is that it can produce several outputs of solution at the off-grid points without requiring additional interpolation. The schemes compare favourable with existing method.

Keywords: Collocation, continuous scheme, interpolation, Laguerre polynomial

Introduction

Linear multistep methods constitute a powerful class of numerical procedures for showing a second order equation of the form

$y'' = f(x,y,y'), y'(a) = z_o, y(a) = y_o, x\epsilon[a.b]$ (1)

It has been well known that an analytical solution to this equation is of little value because many of such problems cannot be solved by analytical approach. In practice, the problems are reduced to systems of first order equations and any method for first order equations are used to solve them. Awoyemi (1999); Fatunla (1998); Lambert (1973) extensively discussed that due to dimension of the problem after it has been reduced to a system of first order equations, the approach waste a lot of computer time and human efforts.

Some attempts has been made to solve problem (1)directly without reduction to a first order systems of equations (Brown, 1977; Lambert, 1991) independently proposed a method known as Multi derivative to solve second order initial value problems type (1) directly. In a recent work of Onumanyi *et al*. (2008), they proposed direct block Adam Moulton Method (BAM) and hybrid block.

Adam Moulton method (IBAM) for accurate approximation to y'appearing in equation (1) to be able to solve problem (1) directly. The aim of this paper is to demonstrate using the present hybrid block method derived to solve equation (1) directly and compare its performance with the block method scheme proposed in Yahaya (2009).

Development of the Method

We set out by approximating the analysis solution of problem (1) with a Laguerre polynomial of the form:

$Y(x) = \sum_{j=o}^{k} a_j\, L_j(x) = y(x)$ (2)

where

$L_{j+1}(x) = (-1)^{n+1} e^x \frac{d^{n+1}}{dx^{n+1}} (e^{-x} x^{n+1})$

so that

$L_o(x) = 1, L_1(x) = (x-1), L_2(x) = x^2 - 4x + 2, L_3(x) = x^3 + 9x^2 + 18x - 6.$

on the partition

$$a = x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_n = b$$

on the integration interval [a, b], with a constant step size h, given by

$$h = x_{n+1} - x_n; n = 0,1, \dots n-1.$$

We need to interpolate at least two points to be able to approximate (1) and, to make this happen, we proceed by arbitrarily selecting an off–step point, x_{n+v},v∈(0,1) in (x_n, x_{n+1}) in such a manner that the zero-stability of the main method is guaranteed. Then (2) is interpolated at $x_{n+i}, i = 0, v$ and its second derivative is collocated at $x_{n+i}, i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ so as to obtain a system of seven equations each of degree six i.e. $k = 6$

$\sum_{j=o}^{6} a_j\, L_j(x) = y(x)$ (3)

$\sum_{j=0}^{6} a_j L_j''(x) = f(x,y,y')$ (4)

Let us arbitrarily set $v = \frac{1}{2}$ then collocating (4) at $x_{n+i}, i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and interpolating (3) at $x_{n+i}, i = 0, \frac{1}{2}$ lead to system of equations of the form;

$$
\begin{aligned}
f_n &= 2a_2 + a_3(6x_n - 18) + a_4(12x_n^2 - 96x_n + 144) + a_5(20x_n^3 - 300x_n^2 + 1800x_n + 1800) \\
&\quad + a_6(30_n^4 - 720x_n^3 + 5400x_n^2 - 14400x_n + 10800) \\
f_{n+1/4} &= 2a_2 + a_3(6x_{n+1/4} - 18) + a_4(12x_{n+1/4}^2 - 96x_{n+1/4} + 144) + a_5(20x_{n+1/4}^3 - 300x_{n+1/4}^2 + 1800x_{n+1/4} + 1800) \\
&\quad + a_6(30_{n+1/4}^4 - 720x_{n+1/4}^3 + 5400x_{n+1/4}^2 - 14400x_{n+1/4} + 10800) \\
f_{n+1/2} &= 2a_2 + a_3(6x_{n+1/2} - 18) + a_4(12x_{n+1/2}^2 - 96x_{n+1/2} + 144) + a_5(20x_{n+1/2}^3 - 300x_{n+1/2}^2 + 1800x_{n+1/2} + 1800) \\
&\quad + a_6(30_{n+1/2}^4 - 720x_{n+1/2}^3 + 5400x_{n+1/2}^2 - 14400x_{n+1/2} + 10800) \\
f_{n+3/4} &= 2a_2 + a_3(6x_{n+3/4} - 18) + a_4(12x_{n+3/4}^2 - 96x_{n+3/4} + 144) + a_5(20x_{n+3/4}^3 - 300x_{n+3/4}^2 + 1800x_{n+3/4} + 1800) \\
&\quad + a_6(30_{n+3/4}^4 - 720x_{n+3/4}^3 + 5400x_{n+3/4}^2 - 14400x_{n+3/4} + 10800) \\
f_{n+1} &= 2a_2 + a_3(6x_{n+1} - 18) + a_4(12x_{n+1}^2 - 96x_{n+1} + 144) + a_5(20x_{n+1}^3 - 300x_{n+1}^2 + 1800x_{n+1} + 1800) \\
&\quad + a_6(30_{n+1}^4 - 720x_{n+1}^3 + 5400x_{n+1}^2 - 14400x_{n+1} + 10800) \\
y_n &= a_0 + a_1(x_n - 1) + a_2(x_n^2 - 4x_n + 2) + a_3(x_n^3 - 9x_n^2 + 18x_n - 6) + a_4(x_n^4 - 16x_n^3 + 72x_n^2 - 96x_n - 24) \\
&\quad + a_5(x_n^5 - 25x_n^4 + 300x_n^3 - 900x_n^2 + 600x_n - 120) \\
&\quad + a_6(x_n^6 - 36x_n^5 + 450x_n^4 - 2400x_n^3 + 5400x_n^2 - 4320x_n + 720)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1/2} = {} & a_0 + a_1(x_{n+1/2} - 1) + a_2(x_{n+1/2}^2 - 4x_{n+1/2} + 2) + a_3(x_{n+1/2}^3 - 9x_{n+1/2}^2 + 18x_{n+1/2} - 6) \\
& + a_4(x_{n+1/2}^4 - 16x_{n+1/2}^3 + 72x_{n+1/2}^2 - 96x_{n+1/2} - 24) \\
& + a_5(x_{n+1/2}^5 - 25x_{n+1/2}^4 + 300x_{n+1/2}^3 - 900x_{n+1/2}^2 + 600x_{n+1/2} - 120) \\
& + a_6(x_{n+1/2}^6 - 36x_{n+1/2}^5 + 450x_{n+1/2}^4 - 2400x_{n+1/2}^3 + 5400x_{n+1/2}^2 - 4320x_{n+1/2} + 720)
\end{aligned}
$$

We solve the system of seven equations by MAPLE to obtain the value of the unknown parameters $a_j, j = 0(1)6$.
Substituting $a_j's$ into (2) yields a continuous implicit hybrid one-step method in the form:

$$Y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h^2[\sum_{j=0}^{1}\beta_j(x)f_{n+j} + \beta_{\frac{1}{4}}(x)\,f_{n+\frac{1}{4}} + \beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}}(x)f_{n+\frac{3}{4}}] \qquad (5)$$

Where $\alpha_j(x) and\ \beta_j(x)$ are continuous coefficient $y_{n+j} = y(x_n + jh)$ is the numerical approximation of the analytical solution at $x_{n+j} and\ f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$.
Equation (5) yields the α_j and β_j as the following continuous function of t:

$$\alpha_o = -1,\ \ \alpha_{1/2} = 2,\ \ \beta_{1/4} = \frac{h^2}{15},\ \ \beta_{1/2} = \frac{13h^2}{120}, \beta_{3/4} = \frac{h^2}{15},\ \beta_1 = \frac{h^2}{240} \qquad (6)$$

Evaluating (5) at x_{n+1}, the main method is obtained as:

$$y_{n+1} + y_n - 2y_{n+1/2} = \frac{h^2}{240}\left[f_n + 16f_{n+\frac{1}{4}} + 26f_{n+\frac{1}{2}} + 16f_{n+\frac{3}{4}} + f_{n+1}\right] \qquad (7)$$

To derive the block method, additional equations are needed since equation (7) alone will not be sufficient for the solution. The additional methods can be obtained by evaluating the first derivative of equation (5):

$$Y'(x) = \frac{1}{h}\left[\alpha'_0(x)y_n + \alpha'_{\frac{1}{2}}(x)y_{n+\frac{1}{2}}\right] + h(\sum_{j=0}^{1}\beta'_j(x)f_{n+j} + \beta'_{\frac{1}{4}}(x)f_{n+\frac{1}{4}} + \beta'_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta'_{\frac{3}{4}}(x)f_{n+\frac{3}{4}}) \qquad (8)$$

at $x_{n+i}, i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ respectively, This yields the following discrete derivative schemes:

$$720hy'_n + 1440y_n - 1440y_{n+1/2} = h^2[-53f_n - 144f_{n+\frac{1}{4}} + 30f_{n+\frac{1}{2}} - 16f_{n+\frac{3}{4}} + 3f_{n+1}] \qquad (9)$$

$$2880hy'_{n+1/4} + 5760y_n - 5760y_{n+1/2} = h^2[39f_n + 70f_{n+\frac{1}{4}} - 144f_{n+\frac{1}{2}} + 42f_{n+\frac{3}{4}} - 7f_{n+1}] \qquad (10)$$

$$720hy'_{n+1/2} + 1440y_n - 1440y_{n+1/2} = h^2[5f_n + 104f_{n+\frac{1}{4}} + 78f_{n+\frac{1}{2}} - 8f_{n+\frac{3}{4}} + f_{n+1}] \qquad (11)$$

$$2880hy'_{n+3/4} + 5760y_n - 5760y_{n+1/2} = h^2[31f_n + 342f_{n+\frac{1}{4}} + 768f_{n+\frac{1}{2}} + 314f_{n+\frac{3}{4}} - 15f_{n+1}] \qquad (12)$$

$$720hy'_{n+1} + 1440y_n - 1440y_{n+1/2} = h^2[3f_n + 112f_{n+\frac{1}{4}} + 126f_{n+\frac{1}{2}} + 240f_{n+\frac{3}{4}} + 59f_{n+1}] \qquad (13)$$

Equations (7), (9), (10), (11), (12) and (13) are solved simultaneously to obtain the following explicit results:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{90}\left[7f_n + 24f_{n+\frac{1}{4}} + 6f_{n+\frac{1}{2}} + 8f_{n+\frac{3}{4}}\right] \qquad (14)$$

$$y_{n+1/2} = y_n + \frac{1}{2}hy'_n + \frac{h^2}{1440}\left[53f_n + 144f_{n+\frac{1}{4}} - 30f_{n+\frac{1}{2}} + 16f_{n+\frac{3}{4}} - 3f_{n+1}\right] \qquad (15)$$

$$y'_{n+1} = y'_n + \frac{h}{90}\left[7f_n + 32f_{n+\frac{1}{4}} + 12f_{n+\frac{1}{2}} + 32f_{n+\frac{3}{4}} + 7f_{n+1}\right] \qquad (16)$$

$$y'_{n+1/2} = y'_n + \frac{h}{360}\left[29f_n + 124f_{n+\frac{1}{4}} + 24f_{n+\frac{1}{2}} + 4f_{n+\frac{3}{4}} - f_{n+1}\right] \qquad (17)$$

$$y'_{n+1/4} = y'_n + \frac{h}{2880}\left[251f_n + 646f_{n+\frac{1}{4}} - 264f_{n+\frac{1}{2}} + 106f_{n+\frac{3}{4}} - 19f_{n+1}\right] \qquad (18)$$

$$y'_{n+3/4} = y'_n + \frac{h}{320}\left[27f_n + 102f_{n+\frac{1}{4}} + 72f_{n+\frac{1}{2}} + 42f_{n+\frac{3}{4}} - 3f_{n+1}\right] \qquad (19)$$

Analysis of the method

The basic properties of the derived Scheme are discussed.
The Explicit Scheme (14-19) derived are discrete Scheme belonging to the class of LMM of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\sum_{j=0}^{k}\beta_j f_{n+j} \qquad (20)$$

The Linear differential operator L defined by

$$L[y(x);h] = \sum_{j=0}^{k}[\alpha_j y(x+jh) - h^2\beta_j y^n(x+jh)] \qquad (21)$$

Expanding (21) by Taylor series, we have

$$L[y(x);h] = C_0y(x) + C_1hy'(x) + \cdots + C_qh^qy^q(x)$$

where

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k$$

$$C_1 = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k$$

$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + \cdots + k^2\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k$$

.
.
.

$$C_p = \frac{1}{p!}(\alpha_1 + 2^p\alpha_2 + \cdots + k^p\alpha_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{p-2}\beta_2 + \cdots + k^{q-2}\beta_k),$$

$q \geq 3$

Order and error constant
Definition 1: The LMM (20) is said to be order P if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$ is the error constant.

Table 1: Showing the Orders and Error Constants of the methods

Equation numbers	Order (P)	Error constants
(14)	5	3.10019841 x 10^{-6}
(15)	5	2.46484953 x 10^{-6}
(16)	5	3.52371964 x 10^{-6}
(17)	5	1.55009921 x 10^{-6}
(18)	5	1.77052330 x 10^{-6}
(19)	5	2.71267361 x10^{-6}

Consistency
Definition 2: The LMM (20) is said to be consistent if it is of order P≥ 1 and its first and second characteristic polynomial defined as $\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$ and $\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$ where Z satisfies $(i) \sum_{j=0}^{k} \alpha_j = 0, (ii) \rho'(1) = 0, (iii)\, \rho''(1) = 2!\,\sigma(1)$, See Lambart (1973).

The discrete Schemes derived are all of order than one and satisfy the condition (i)-(iii)

Zero Stability of the block method
The block method is defined by Fatunla (1988) as

$$Y_m = \sum_{i=0}^{k} A_i + h \sum_{i=0}^{k} B_i\, F_{m-i}$$

$where\ Y_m = [y_n, y_{n+1}, y_{n+2}, \dots, y_{n+r-1}]^{\mathrm{T}}$
$F_m = [f_n, f_{n+1}, f_{n+2}, \dots, f_{n+r-1}]^{\mathrm{T}}$
$A_i's$ and $B_i's$ are chosen r x r matrix coefficient and $m = 0,1,2 \ldots$ represents the block number, $n = mr$, the first step number in the m-th block and r is the proposed block size.
The block method is said to be zero stable if the roots of $R_j, j = 1(1)k$ of the first characteristics polynomial is

$$\rho(R) = \det\left[\sum_{i=0}^{k} A_i R^{k-1}\right] = 0, A_0 = I$$

satisfies $|\mathbf{R}_j| \leq \mathbf{1}$, if one of the roots is +1, then the root is called Principal Root of $\rho(R)$.

Zero-stability for schemes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y'_{n+1} \\ y'_{n+1/2} \\ y'_{n+1/4} \\ y'_{n+3/4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y'_{n-1/4} \\ y'_{n-1/2} \\ y'_{n-1} \\ y'_{n} \end{bmatrix} + h \begin{bmatrix} \frac{32}{90} & \frac{12}{90} & \frac{32}{90} & \frac{7}{90} \\ \frac{124}{360} & \frac{24}{360} & \frac{4}{360} & \frac{-1}{360} \\ \frac{646}{2880} & \frac{-264}{2880} & \frac{106}{2880} & \frac{-19}{2880} \\ \frac{102}{320} & \frac{72}{320} & \frac{42}{320} & \frac{-3}{320} \end{bmatrix} \begin{bmatrix} f_{n+1/4} \\ f_{n+1/2} \\ f_{n+3/4} \\ f_{n+1} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & \frac{7}{90} \\ 0 & 0 & 0 & \frac{29}{360} \\ 0 & 0 & 0 & \frac{251}{2880} \\ 0 & 0 & 0 & \frac{27}{320} \end{bmatrix} \begin{bmatrix} f_{n-3/4} \\ f_{n-1/2} \\ f_{n-1/4} \\ f_{n} \end{bmatrix}$$

where

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B^{(0)} = \begin{bmatrix} \frac{32}{90} & \frac{12}{90} & \frac{32}{90} & \frac{7}{90} \\ \frac{124}{360} & \frac{24}{360} & \frac{4}{360} & \frac{-1}{360} \\ \frac{646}{2880} & \frac{-264}{2880} & \frac{106}{2880} & \frac{-19}{2880} \\ \frac{102}{320} & \frac{72}{320} & \frac{42}{320} & \frac{-3}{320} \end{bmatrix} \text{ and}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{7}{90} \\ 0 & 0 & 0 & \frac{29}{360} \\ 0 & 0 & 0 & \frac{251}{2880} \\ 0 & 0 & 0 & \frac{27}{320} \end{bmatrix}$$

The first characteristics polynomial of the scheme is
$\rho(\lambda) = \det[\lambda A^0 - A^1]$

$$\rho(\lambda) = det\left[\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$\rho(\lambda) = det\begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = 0$$

$\lambda^3(\lambda - 1) = 0$
$\lambda_1 = \lambda_2 = \lambda_3 = 0 \; or \; \lambda_4 = 1$

We can see clearly that no root has modulus greater than one (i.e. $\lambda_i \leq 1) \forall i$. The hybrid block method is zero stable.

Numerical experiment
$y'' - y' = 0, \qquad y(0) = 0, \qquad y'(0) = -1, \;\; h = 0.1$
Exact Solution:$y(x) = 1 - \exp(x)$

Table 2: The exact solution and the computed results from the proposed method two for problem 1

x	Exact Solution	New Method	Yahaya 2009	Error in New method	Error in Yahaya 2009
0.1	-0.105170918	-0.1051709181	-0.105170902	9.999999E-11	0.160756E-07
0.2	-0.221402758	-0.2214027582	-0.221402723	0.245218E-09	0.351602E-07
0.3	-0.349858807	-0.3498588077	-0.34985857	0.734286E-09	0.237576E-06
0.4	-0.491824697	-0.4918246978	-0.491824433	0.835326E-09	0.2646413E-06
0.5	-0.64872127	-0.6487212709	-0.648720974	0.945324E-09	0.2967001E-06
0.6	-0.82211880	-0.8221188007	-0.822118466	0.734287E-09	0.3343905E-06
0.7	-1.013752707	-1.013752708	-1.013752329	0.193453E-08	0.3784705E-06
0.8	-1.225540928	-1.225540929	-1.225540498	0.156723E-08	0.4304925E-06
0.9	-1.459603111	-1.459603112	-1.45960262	0.165782E-08	0.4911569E-06
1.0	-1.718281828	-1.718281830	-1.718281267	0.224176E-08	0.561459E-06

Conclusion
In this paper, it is observed from the table that the result obtained from the method converged faster when the numbers of off-step points were increased. This validates the consistency and zero stability of the methods. Generally, the performance of our method as noticed in **Table 2** shows that the proposed method is more superior to the block methods proposed by Yahaya.

References

Awoyemi DO 1999. A class of continuous methods for general second order initial value problems in ordinary differential equations. *Int. J. Computational Maths.*, 72: 29-37.

Badmus AM 2006. A reformulation of P-stable Schemes in continuous form for second order initial value problems in ordinary differential equations. M.Sc Thesis Abubakar Tafawa Balewa University Bauchi, Nigeria.

Brown RL 1977. Some characteristics of implicitly multi derivative integration formulas. *SIAM. J. Numer. Anal*. 14: 982-983.

Fatunla SO 1988. Numerical Methods for Initial Value Problems for Ordinary Differential Equations. USA Academy Press, Boston, p. 295.

Lanbert JD 1991. Numerical Methods for Ordinary Differential Systems. New York, John Willey.

Onumanyi P, Fatokun J & Adejo BO 2008. Accurate Numerical Differentiation by continuous integration for ordinary differential equations. *J. Maths. Assoc. Nig.*, 27: 69-90.

Yahaya YA 2009. Some Theories and Application of Continuous Linear Multistep Method for ordinary differential equations PhD Thesis University of Jos, Nigeria.